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LETTER TO THE EDITOR

Bicomplexes and conservation laws in non-Abelian Toda models

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Abstract
A bicomplex structure is associated with the Leznov–Saveliev equation of integrable models. The linear problem associated with the zero-curvature condition is derived in terms of the bicomplex linear equation. The explicit example of a non-Abelian conformal affine Toda model is discussed in detail and its conservation laws are derived from the zero-curvature representation of its equation of motion.

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1. Introduction

Two-dimensional Toda field theories are examples of relativistic integrable nonlinear systems underlined by a Lie algebraic structure. Finite-dimensional Lie algebras are associated with the conformal Toda models (see [1] for a review), whose basic representative is the Liouville model. The affine Toda models are associated with the loop algebra (centreless Kac–Moody) and are characterized by broken conformal symmetry. As basic examples within this class, we find the sine–Gordon, Lund–Regge (complex sine–Gordon), general Abelian affine Toda and homogeneous sine–Gordon [2]. Dyonic integrable models such as the singular non-Abelian Toda models are also within this class [3]. Conformal affine Toda models [4] are related to infinite-dimensional affine lie algebras (full Kac–Moody algebras). Such models are classified according to a grading operator decomposing the Lie algebra into graded subspaces. The graded structure is an important ingredient to obtain such models when employing the Hamiltonian reduction procedure to the WZNW [1] and two-loop WZNW models [4]. Alternatively, the field equations of these models can be obtained from the Leznov–Saveliev equation [5]. An important fact about this equation is that it can be written as a zero-curvature condition. As a consequence, under specific boundary conditions, there are infinite conserved charges. Also, if the fundamental Poisson bracket relation holds [6], the involution condition among the conserved charges is verified. An important ingredient in this construction is the classical r-matrix satisfying the classical Yang–Baxter equation.

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In recent papers [7], a structure called bicomplex was used to derive some integrable field equations (e.g. sine–Gordon, nonlinear Schrödinger). It was argued that the bicomplex linear equation could, in some cases, lead to chains of conserved charges.

In this paper we generalize the bicomplex structure to derive the Leznov–Saveliev equation corresponding to an infinite-dimensional affine Lie algebra, which includes the non-Abelian Toda equations. The linear problem associated with the zero-curvature condition is also derived in terms of the bicomplex linear equation. Explicit construction, following the arguments of [8], for the conserved charges of a specific $A_2^{(1)}$ non-Abelian Toda model is obtained.

2. Bicomplexes and Leznov–Saveliev equation

Let $V = \oplus_{r\geqslant 0} V^r$ be an N_0-graded linear space over C and $d, \delta : M^r \to M^{r+1}$ linear maps. If $d^2 = \delta^2 = \delta d + d\delta = 0$, then this structure is called a bicomplex [7].

It is important to emphasize that nothing is said about Leibnitz rules. Let ξ^1, ξ^2 be a basis for V^1 such that $\xi^1\xi^1 = \xi^2\xi^2 = \xi^1\xi^2 + \xi^2\xi^1 = 0$. In this case V^2 is one dimensional and $V = V^0 \oplus V^1 \oplus V^2$. It is convenient to introduce light-cone variables in the two-dimensional space–time with coordinates (t, x): $z = (t + x)/2$; $\bar{z} = (t - x)/2$; $\partial = \partial/\partial z = \partial_t + \partial_x$; $\bar\partial = \partial/\partial\bar{z} = \partial_t - \partial_x$. Consider a infinite-dimensional affine Lie algebra [9] $\hat{G}$ and constant generators $(\varepsilon^+, \varepsilon^-) \in \hat{G}$ such that

$$[\varepsilon^+, \varepsilon^-] = \mu_1\mu_2\hat{C} \tag{2.1}$$

where $\hat{C}$ is the central charge generator and $(\mu_1, \mu_2) \in C$. The meaning of this choice will be explained at the end of this section.

Let $v^1 = (v^1_1\xi^1 + v^1_2\xi^2) \in V^1$ be arbitrary and define

$$\delta v^1 \equiv (\delta v^1_1)\xi^1 + (\delta v^1_2)\xi^2 \qquad dv^1 \equiv (dv^1_1)\xi^1 + (dv^1_2)\xi^2. \tag{2.2}$$

Similarly, for $v^2 = v^2_{1,2}\xi^1\xi^2 \in V^2$ arbitrary, define

$$\delta v^2 \equiv (\delta v^2_{1,2})\xi^1\xi^2 = 0 \qquad dv^2 \equiv (dv^2_{1,2})\xi^1\xi^2 = 0. \tag{2.3}$$

Let $v^0 \in V^0$ be arbitrary and define the maps δ, d:

$$\delta v^0 \equiv \bar\partial v^0\xi^1 + \varepsilon^- v^0\xi^2 \qquad dv^0 \equiv -\varepsilon^+ v^0\xi^1 + \partial v^0\xi^2. \tag{2.4}$$

An explicit computation reveals that for $v^0 \in V^0$ arbitrary

$$\begin{aligned}\delta^2 v^0 &= \delta(\bar\partial v^0)\xi^1 + \delta(\varepsilon^- v^0)\xi^2\\ &= \bar\partial(\bar\partial v^0)(\xi^1)^2 + \varepsilon^-\bar\partial v^0\xi^2\xi^1 + \bar\partial(\varepsilon^- v^0)\xi^1\xi^2 + (\varepsilon^-)^2 v^0(\xi^2)^2\\ &= \varepsilon^-\bar\partial v^0(\xi^2\xi^1 + \xi^1\xi^2) = 0.\end{aligned}$$

That is,

$$\delta^2 = 0 \qquad d^2 = 0 \qquad (\delta d + d\delta)v^0 = -\mu_1\mu_2\hat{C}v^0\xi^1\xi^2 \tag{2.5}$$

where the last two equations are derived in a similar way. The last equality can be rewritten as

$$P^2 + (\delta d + d\delta) = 0 \tag{2.6}$$

where the map $P : V^r \to V^{r+1}$ is defined by

$$Pv^0 \equiv \varepsilon^+ v^0\xi^1 + \varepsilon^- v^0\xi^2.$$

The action of P in V^1 and V^2 is defined in the same way as done for d, δ. Notice that the maps (δ, d) do not define a bicomplex, unless the central charge is taken equal to zero, which implies that we are working with the loop algebra. Alternatively, let g be an exponential of the generators belonging to $\hat{G}$. Define a dressing [7] for d, introducing $D : V^r \to V^{r+1}$ such that, for arbitrary $v^0 \in V^0$

$$Dv^0 \equiv g^{-1}d(gv^0) = -g^{-1}\varepsilon^+ gv^0\xi^1 + (\partial + g^{-1}\partial g)v^0\xi^2. \tag{2.7}$$

Extending the action of D in V^1 and V^2 in the same way as before,

$$D^2v^0 = g^{-1}d(gDv^0) = g^{-1}d(gg^{-1}d(gv^0)) = 0 \;\to\; D^2 = 0 \tag{2.8}$$

using (2.5) and the fact that v^0 is arbitrary. Now,

$$(\delta D + D\delta)v^0 = \{\bar{\partial}(g^{-1}\partial g) - [g^{-1}\varepsilon^+ g, \varepsilon^-]\}v^0\xi^1\xi^2. \tag{2.9}$$

In order to obtain the Leznov–Saveliev equation there are two different options here. The first is to take

$$g = B\exp(-\mu_1\mu_2 z\bar{z}\hat{C}) \tag{2.10}$$

where B is a group element and impose

$$\delta D + D\delta = \delta d + d\delta = -P^2. \tag{2.11}$$

As a consequence,

$$\bar{\partial}(B^{-1}\partial B) = [B^{-1}\varepsilon^+ B, \varepsilon^-] \qquad \partial(\bar{\partial}BB^{-1}) = [\varepsilon^+, B\varepsilon^- B^{-1}]. \tag{2.12}$$

Equations (2.12) correspond to the Leznov–Saveliev equation [5] in its two different versions. Notice, however, that the maps (δ, D) defined in this way do not define a bicomplex.

Consider now the second option. Take

$$g = B \;\to\; Dv^0 \equiv B^{-1}d(Bv^0) = -B^{-1}\varepsilon^+ Bv^0\xi^1 + (\partial + B^{-1}\partial B)v^0\xi^2 \tag{2.13}$$

for $v^0 \in V^0$ arbitrary. Extend the action in V^1 and V^2 as before and impose $\delta D + D\delta = 0$. This leads to the Leznov–Saveliev equation again and, in this case, defines a bicomplex:

$$D^2 = \delta^2 = \delta D + D\delta = 0. \tag{2.14}$$

An explanation of (2.1) is important. If $(\varepsilon^+, \varepsilon^-)$ are chosen in such a way that (2.1) holds, then $B_0 = \exp(\mu_1\mu_2 z\bar{z}\hat{C})$ is a particular solution of the Leznov–Saveliev equation (2.12). In fact, this is a vacuum solution required by the dressing method as an input for non-trivial one soliton solutions [3].

3. The bicomplex linear equation

A linear problem is associated with a given bicomplex in the following way [7]: suppose there is $T^{(0)} \in V^0$ such that $DJ^{(0)} = 0$, where $J^{(0)} \equiv \delta T^{(0)}$. Using (2.14), $\delta J^{(0)} = 0$. Defining $J^{(1)} \equiv DT^{(0)}$ and using (2.14), $\delta J^{(1)} = 0$, $DJ^{(1)} = 0$. Suppose that $J^{(1)}$ can also be written as $J^{(1)} = \delta T^{(1)}$, $T^{(1)} \in V^{(0)}$. Then, defining $J^{(2)} \equiv DT^{(1)}$, $\delta J^{(2)} = -D\delta T^{(1)} = -DJ^{(1)} = 0$ and $DJ^{(2)} = 0$. Continuing indefinitely such steps and defining a formal expansion $T \equiv \sum_{m=0}^{\infty} \rho^m T^{(m)}$, $\rho \in C$, the bicomplex linear equation is obtained:

$$\delta(T - T^{(0)}) = \rho DT \;\to\; \delta T = \rho DT \tag{3.1}$$

if $\delta T^{(0)} = 0$. Using (2.4) and (2.13) in (3.1) results in

$$\bar{\partial}T = -\bar{A}T \qquad \partial T = -AT \qquad A = -\rho^{-1}\varepsilon^- + B^{-1}\partial B \quad \bar{A} = \rho B^{-1}\varepsilon^+ B. \tag{3.2}$$

Defining

$$\tilde{\varepsilon^{\pm}} = \rho^{\pm 1}\varepsilon^{\pm} \qquad [\tilde{\varepsilon^+}, \tilde{\varepsilon^-}] = [\varepsilon^+, \varepsilon^-] = \mu_1\mu_2\hat{C} \tag{3.3}$$

the commutation relation acquires the same structure. In fact, even the Leznov–Saveliev equation is invariant under (3.3). Now,

$$A = -\tilde{\varepsilon^-} + B^{-1}\partial B \qquad \bar{A} = B^{-1}\tilde{\varepsilon^+}B \qquad \partial\bar{A} - \bar{\partial}A + [A, \bar{A}] = 0 \tag{3.4}$$

is a standard representation of the connections associated with the zero-curvature equation from which the Leznov–Saveliev equation is derived. The solution of (3.2) is [6]

$$T(t, y) = T_0\mathcal{P}\left[\exp\left(\int^{(t,y)} A_\mu \, \mathrm{d}x^\mu\right)\right] \tag{3.5}$$

where $\mathcal{P}$ is the path ordered operator and T_0 is a constant. We should point out that the spectral parameter ρ naturally arises under such a framework in (3.2). Given an affine Lie algebra $\hat{G}$ and a grading operator $\hat{Q} \in \hat{G}$ follows a decomposition [10]: $\hat{G} = \oplus_i \hat{G}_i$; $[\hat{Q}, \hat{G}_i] = \mathrm{i}\hat{G}_i$; $[\hat{G}_i, \hat{G}_j] \in \hat{G}_{i+j}$. Here, $i \in Z$. The Hamiltonian reduction procedure applied to the two-loop WZNW model [4] leads to the Leznov–Saveliev equation, where the group element B is associated with zero-grade generators of $\hat{G}$ and $\varepsilon\pm$ are generators of grade $\pm j$, $j \in Z$. That is, these integrable models are classified in terms of the grading operators [10]. In particular, the class of singular $A_n^{(1)}$ non-Abelian Toda models has been constructed in [3] by choosing the zero-grade subgroup $\hat{\mathcal{G}}_0 = SL(2) \otimes U(1)^{n-1}$. One can realize (3.3) as

$$\tilde{\varepsilon^\pm} = \exp\left(\frac{\hat{Q}\ln\rho}{j}\right)\varepsilon^\pm \exp\left(\frac{-\hat{Q}\ln\rho}{j}\right). \tag{3.6}$$

4. Conformal affine non-Abelian Toda model

In this section we consider the example of a $A_2^{(1)}$ conformal affine non-Abelian Toda model, whose singular version was constructed in [3]. The zero-grade subgroup $\hat{\mathcal{G}}_0 = SL(2)\otimes U(1) \subset SL(3)$ is parametrized by

$$B = \exp(\beta\tilde{\chi}E^{(0)}_{-\alpha_1})\exp(\beta\varphi_1 H^{(0)}_{\lambda_1} + \beta\varphi_2 h^{(0)}_2 + \beta\nu\hat{C} + \beta\eta\hat{D})\exp(\beta\tilde{\psi}E^{(0)}_{\alpha_1}) \tag{4.1}$$

$$\hat{Q} = 2\hat{D} + H^{(0)}_{\lambda_2} \qquad \varepsilon^+ = \mu_1(E^{(0)}_{\alpha_2} + E^{(1)}_{-\alpha_2}) \qquad \varepsilon^- = \mu_2(E^{(0)}_{-\alpha_2} + E^{(-1)}_{\alpha_2}) \tag{4.2}$$

where $\hat{D}$ is the homogeneous grading operator, $h_i^{(0)} = 2\alpha_i.H^{(0)}/\alpha_i^2$ are Chevalley generators, $H_i^{(0)}$ define the A_2 Cartan subalgebra in the Weyl–Cartan basis, $H^{(0)}_{\lambda_i} = 2\lambda_i.H^{(0)}/\alpha_i^2$, λ_i are the fundamental weights of A_2 satisfying $2\alpha_i \cdot \lambda_j/\alpha_i^2 = \delta_{i,j}$, that is, $\lambda_i = \sum_{j=1}^2 (K^{-1})_{i,j}\alpha_j$, K is the Cartan matrix of A_2, $(i, j) = (1, 2)$, $\beta^2 = -\beta_0^2 = -(2\pi)/k$ and k is the WZNW coupling constant. Also, the normalization $\alpha^2 = 2$ for all the roots is adopted (see [9] for a description of affine Lie algebras).

The constant generators $\varepsilon^\pm$ have grade ± 1 with respect to the generalized grading operator $\hat{Q}$ and $B \in \hat{\mathcal{G}}_0$. This grading is an intermediate between the homogeneous grading $\hat{Q} = \hat{D}$ and the principal grading $\hat{Q} = 3\hat{D} + H^{(0)}_{\lambda_1} + H^{(0)}_{\lambda_2}$ [10]. The Leznov–Saveliev equation leads to the field equations corresponding to the Lagrangian density

$$\begin{aligned}\mathcal{L} = {} & (1/3)\partial\varphi_1\bar{\partial}\varphi_1 + \partial\varphi_2\bar{\partial}\varphi_2 + (1/2)(\partial\nu\bar{\partial}\eta + \partial\eta\bar{\partial}\nu) + \exp(\beta(\varphi_1 - \varphi_2))\partial\tilde{\chi}\bar{\partial}\tilde{\psi} \\ & -(\mu_1\mu_2/\beta^2)(\exp(-2\beta\varphi_2) + \exp(\beta(2\varphi_2 - \eta)) + \beta^2\tilde{\psi}\tilde{\chi}\exp(\beta(\varphi_1 + \varphi_2 - \eta))).\end{aligned} \tag{4.3}$$

In order to construct the singular non-Abelian Toda model, one observes that $\hat{G}_0^0 \equiv H^{(0)}_{\lambda_1} \in \hat{G}_0$ is such that $[\hat{G}_0^0, \epsilon_\pm] = 0$, implying, from the Leznov–Saveliev equation, the chiral conservation laws $\partial\,\mathrm{Tr}[\hat{G}_0^0\bar{\partial}BB^{-1}] = \bar{\partial}\,\mathrm{Tr}[\hat{G}_0^0 B^{-1}\partial B] = 0$. These, in turn, allow the subsidiary constraints $\mathrm{Tr}[\hat{G}_0^0 \bar{\partial} B B^{-1}] = \mathrm{Tr}[\hat{G}_0^0 B^{-1} \partial B] = 0$, responsible for the elimination of the non-local field φ_1, $\partial\varphi_1 = \frac{3}{2}\frac{\beta\psi\partial\chi\exp(-\beta\varphi_2)}{\Delta}$; $\bar{\partial}\varphi_1 = \frac{3}{2}\frac{\beta\chi\bar{\partial}\psi\exp(-\beta\varphi_2)}{\Delta}$, where $\chi = \tilde{\chi}\exp(\beta\varphi_1/2)$; $\psi = \tilde{\psi}\exp(\beta\varphi_1/2)$; $\Delta = 1 + (3/4)\beta^2\psi\chi\exp(-\beta\varphi_2)$. The classical r-matrix associated with this model is discussed in [11]. The singular non-Abelian affine Toda model, that is, without the field η (the field ν is only an auxiliary field) has already been discussed in the literature. In [3] a complete spectrum of one- and two-soliton solutions was obtained using the dressing transformations and the vertex operator construction. Also, the T-dual version of this model was analysed and some results on semiclassical quantization were shown.

5. Conservation laws

In order to derive the conservation laws for the model defined in (4.3), it is convenient to define a new basis for $A_2^{(1)}$:

$$\begin{array}{llll} \hat{Q} = 2\hat{D} + H_{\lambda_2}^{(0)} & \hat{C} \quad A_{(2n)} = \sqrt{3}\big(H_{\lambda_1}^{(n)} - (1/6)\delta_{n,0}\hat{C}\big) & & A_{(2n+1)} = E_{\alpha_2}^{(n)} + E_{-\alpha_2}^{(n+1)} \\ F_{(2n)} = h_2^{(n)} - (1/2)\delta_{n,0}\hat{C} & F_{(2n+1)} = E_{\alpha_2}^{(n)} - E_{-\alpha_2}^{(n+1)} & & \\ F_{(2n)}^{+} = E_{\alpha_1}^{(n)} & F_{(2n+1)}^{+} = E_{\alpha_1+\alpha_2}^{(n)} & F_{(2n)}^{-} = E_{-\alpha_1}^{(n)} & F_{(2n-1)}^{-} = E_{-\alpha_1-\alpha_2}^{(n)} \end{array} \tag{5.1}$$

where $n \in Z$. The generators $A_{(2n)}, A_{(2n+1)}$ define a infinite-dimensional Heisenberg subalgebra:

$$\begin{array}{ll} [A_{(2n)}, A_{(2m+1)}] = 0 & [A_{(2n)}, A_{(2m)}] = 2n\delta_{n+m,0} \\ [A_{(2n+1)}, A_{(2m+1)}] = (2n+1)\delta_{n+m+1,0}. & \end{array} \tag{5.2}$$

Also, it is verified that

$$\begin{array}{ll} [F_{(2n+1)}, A_{(2m)}] = [F_{(2n)}, A_{(2m)}] = 0 & \\ [F_{(2n+p)}^{+}, A_{(2m)}] = -\sqrt{3}F_{[2(n+m)+p]}^{+} & [F_{(2n-p)}^{-}, A_{(2m)}] = \sqrt{3}F_{[2(n+m)-p]}^{-} \\ [F_{(2n+p)}^{+}, A_{(2m+1)}] = F_{[2(n+m+p)+1-p]}^{+} & [F_{(2n-p)}^{-}, A_{(2m+1)}] = -F_{[2(n+m+1-p)+p-1]}^{-} \\ [F_{(2n)}, A_{(2m+1)}] = 2F_{[2(n+m)+1]} & [F_{(2n+1)}, A_{(2m+1)}] = 2F_{[2(n+m+1)]} \end{array} \tag{5.3}$$

where $p = (0, 1)$. Defining $\mathcal{A} = \{A_{(2n+p)}, \hat{Q}, \hat{C}\}$ and $\mathcal{F} = \{F_{(2n+p)}, F_{(2n\pm p)}^{\pm}\}$, we see that linear combinations of generators $\in \mathcal{F}$ are used to construct the vertex operators [10], used in the dressing method [3].

The conservation laws follow from the zero-curvature equation by gauge transforming A and $\bar{A}$ into A_{ab}^{R} and $\bar{A}_{ab}^{R}$ such that $[A_{ab}^{R}, \bar{A}_{ab}^{R}] = 0$ [8, 12]. It is convenient to define the notation $\mathcal{F} = \mathcal{F}^{+} \oplus \mathcal{F}^{-} \oplus \mathcal{F}^{0}$; $\quad \mathcal{A} = \mathcal{A}^{+} \oplus \mathcal{A}^{-} \oplus \mathcal{A}^{0}$, where the subspaces $(\mathcal{F}^{\pm}, \mathcal{F}^{0})$ have a (positive/negative, zero) grade w.r.t. $\hat{Q}$ and similarly to $(\mathcal{A}^{\pm}, \mathcal{A}^{0})$. Consider

$$A = B\varepsilon^{-}B^{-1} \qquad \bar{A} = -\varepsilon^{+} - \bar{\partial}BB^{-1} \qquad g_R = \left[\prod_{m=1}^{\infty}\exp(S_{-m})\right]\exp(\xi\varepsilon^{-}) \tag{5.4}$$

where the connections result in the Leznov–Saveliev equation under the zero-curvature equation and S_{-m} is a linear combination of generators in $\mathcal{F}^{-}$ and $\xi(z, \bar{z})$ is a function of z and $\bar{z}$. Consider the gauge transformation:

$$\begin{aligned} A^{R} &= g_R A g_R^{-1} - \partial g_R g_R^{-1} = \sum_{m=-\infty}^{-1}(A_{\mathcal{A}}^{R,m} + A_{\mathcal{F}}^{R,m}) \\ \bar{A}^{R} &= g_R \bar{A} g_R^{-1} - \bar{\partial} g_R g_R^{-1} = \sum_{m=-\infty}^{1}\bar{A}^{R,m} \end{aligned} \tag{5.5}$$

where $\bar{A}^{R,m} \in \hat{G}_m$, $A^{R,m}_{\mathcal{A}}$ has grade m and is a linear combination of generators in $\mathcal{A}$ and similarly for $A^{R,m}_{\mathcal{F}}$. Explicitly,

$$\begin{aligned}
&\bar{A}^{R,1} = -\varepsilon^{+} \qquad \bar{A}^{R,0} = -\bar{\partial} B B^{-1} + \mu_1\mu_2\xi\hat{C} - [S_{-1}, \varepsilon^{+}] \\
&\bar{A}^{R,-1} = -\bar{\partial}(S_{-1} + \xi\varepsilon^{-}) - [S_{-2}, \varepsilon^{+}] - (1/2)[S_{-1}, [S_{-1}, \varepsilon^{+}]] \\
&\qquad\qquad -[\xi\varepsilon^{-} + S_{-1}, \bar{\partial} B B^{-1}] \ldots
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
\bar{\partial} B B^{-1} &= \beta\big[\bar{\partial}\varphi_1 - (3/2)\beta\tilde{\chi}\bar{\partial}\tilde{\psi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}\big]H^{(0)}_{\lambda_1} + \beta\big[\bar{\partial}\varphi_2 + (1/2)\beta\tilde{\chi}\bar{\partial}\tilde{\psi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}\big]h^{(0)}_2 \\
&\quad +\beta[\bar{\partial}\nu\hat{C} + \bar{\partial}\eta\hat{D}] + \beta\bar{\partial}\tilde{\psi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}E^{(0)}_{\alpha_1} \\
&\quad +\beta\big[\bar{\partial}\tilde{\chi} + \beta\tilde{\chi}\bar{\partial}(\varphi_1-\varphi_2) - \beta^2\tilde{\chi}^2\bar{\partial}\tilde{\psi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}\big]E^{(0)}_{-\alpha_1} \\
&\equiv \bar{J}^{(1)}H^{(0)}_{\lambda_1} + \bar{J}^{(2)}h^{(0)}_2 + \bar{J}^{\nu}\hat{C} + \bar{J}^{\eta}\hat{D} + \bar{J}^{+}E^{(0)}_{\alpha_1} + \bar{J}^{-}E^{(0)}_{-\alpha_1}.
\end{aligned} \tag{5.7}$$

Now, choose $\{S_{-m}\}$ and ξ such that

$$\begin{aligned}
&\bar{A}^{R} = -\varepsilon^{+} - \bar{J}^{\eta}\hat{D}_{\mathcal{A}} - \bar{J}^{(1)}H^{(0)}_{\lambda_1} + \sum_{m=-\infty}^{-1}\bar{a}^{R,m}A_{(m)} \\
&\hat{D} \equiv \hat{D}_{\mathcal{A}} + \hat{D}_{\mathcal{F}} = \left[(1/2)\hat{Q} - \frac{\sqrt{3}}{12}A_{(0)} - (1/6)\hat{C}\right] + \left[-\frac{F_{(0)}}{4}\right].
\end{aligned} \tag{5.8}$$

The idea behind this structure is to solve for S_{-m} such that all terms in $\mathcal{F}$ are eliminated. Consider the zero-curvature equation and $(A^R, \bar{A}^R)$ as described. Note that $A^{R,m}_{\mathcal{A}} = a^{R,m}_{\mathcal{A}}A_{(m)}$. In terms of the new basis (5.1), we find

$$\begin{aligned}
&\partial\bar{J}^{\eta} = 0 \qquad \partial\bar{J}^{(1)} = 0 \qquad a^{R,-1}_{\mathcal{A}} = 0 \qquad \partial\bar{a}^{R,-1} = 0 \\
&\partial\bar{a}^{R,-m} - \bar{\partial}a^{R,-m}_{\mathcal{A}} - \frac{m}{2}\bar{J}^{\eta}a^{R,-m}_{\mathcal{A}} = 0 \qquad m \geqslant 2 \\
&A^{R,m}_{\mathcal{F}} = 0 \qquad \forall m
\end{aligned} \tag{5.9}$$

where $\bar{J}^{\eta}$ and $\bar{J}^{(1)}$ are defined in (5.7). Under a gauge transformation defined by the group element g^R_R there follows

$$g^R_R = \exp\left[-\varepsilon^{+}\int_{\bar{L}}^{\bar{z}}\mathrm{e}^{-\frac{1}{2}\int_{\bar{L}}^{\bar{w}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v}}\mathrm{d}\bar{w}\right]\exp\left[-\int_{\bar{L}}^{\bar{z}}(\bar{J}^{(1)}(\bar{w})H^{(0)}_{\lambda_1} + \bar{J}^{\eta}(\bar{w})\hat{D}_{\mathcal{A}})\,\mathrm{d}\bar{w}\right] \tag{5.10}$$

$$\partial\bar{A}^R_{ab} - \bar{\partial}A^R_{ab} = 0 \tag{5.11}$$

where

$$\begin{aligned}
A^R_{ab} &\equiv \sum_{m=-\infty}^{-1}A^{R,m}_{ab} = \sum_{m=-\infty}^{-1}a^{R,m}_{\mathcal{A}}\mathrm{e}^{-\frac{m}{2}\int_{\bar{L}}^{\bar{z}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v}}A_{(m)} \\
\bar{A}^R_{ab} &\equiv \sum_{m=-\infty}^{-1}\bar{A}^{R,m}_{ab} + a^{\hat{C}}\hat{C} \\
&= \sum_{m=-\infty}^{-1}\bar{a}^{R,m}\mathrm{e}^{-\frac{m}{2}\int_{\bar{L}}^{\bar{z}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v}}A_{(m)} - \mu_1\bar{a}^{R,-1}\mathrm{e}^{\frac{1}{2}\int_{\bar{L}}^{\bar{z}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v}}\int_{\bar{L}}^{\bar{z}}\mathrm{e}^{-\frac{1}{2}\int_{\bar{L}}^{\bar{w}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v}}\,\mathrm{d}\bar{w}\,\hat{C}
\end{aligned} \tag{5.12}$$

where $\bar{L} \in R$. Taking $\eta(z,\bar{z}) = \eta_1(z) + \eta_2(\bar{z})$ as a solution for the equation of motion for η results in $\int_{\bar{L}}^{\bar{z}}\bar{J}^{\eta}(\bar{v})\,\mathrm{d}\bar{v} = \eta_2(\bar{z}) - \eta_2(\bar{L})$. This implies that all the terms in the Abelian connections are local, except the term in $\hat{C}$. Under periodic boundary conditions [8, 12], the zero-curvature equation for the Abelianized connections implies an infinite set of conserved charges:

$$\begin{aligned}
&\partial_t Q^R_m = 0 \qquad m \leqslant -1 \\
&Q^R_m = \int_{-s}^{s}A^{R,m}_{x,ab}(t,y)\,\mathrm{d}y \qquad A^{R,m}_{x,ab} = \tfrac{1}{2}(A^{R,m}_{ab} - \bar{A}^{R,m}_{ab})
\end{aligned} \tag{5.13}$$

where $s \in R$. In order to verify the involution condition, one starts from the fundamental Poisson bracket relation (see for instance [11]):

$$
\begin{aligned}
&\{A_x^S(y,t) \otimes A_x^S(z,t)\}_{\mathrm{PB}} = [r, A_x^S(y,t) \otimes I + I \otimes A_x^S(z,t)]\delta(y-z) \\
&A_x^S = \tfrac{1}{2} S(A - \bar{A})S^{-1} - \partial_x S S^{-1} \qquad S = \mathrm{e}^{-\frac{1}{2}(\beta\varphi_1 H_{\lambda_1}^{(0)} + \beta\varphi_2 h_2^{(0)} + \beta\nu\hat{C} + \beta\eta\hat{D})} \mathrm{e}^{-\beta\tilde{\chi} E_{-\alpha_1}^{(0)}} \\
&r = \frac{\beta^2}{4}(C^+ - \sigma C^+) \\
&\sigma(a \otimes b) = b \otimes a \qquad \forall (a,b) \in \hat{G} \\
&C^+ = \sum_{m=1}^{\infty} \sum_{a,b=1}^{2} \frac{\alpha_b^2}{2} (K^{-1})_{a,b} \left(h_a^{(m)} \otimes h_b^{(-m)} \right) + \frac{1}{2} \sum_{\alpha>0} \frac{\alpha^2}{2} \left(E_\alpha^{(0)} \otimes E_{-\alpha}^{(0)} \right) \\
&\qquad + \sum_{m=1}^{\infty} \sum_{\alpha>0} \frac{\alpha^2}{2} \left[E_\alpha^{(m)} \otimes E_{-\alpha}^{(-m)} + E_{-\alpha}^{(m)} \otimes E_\alpha^{(-m)} \right]
\end{aligned}
\tag{5.14}
$$

where $(A, \bar{A})$ are defined in (5.4). As a consequence [8, 12],

$$
\{\mathrm{tr}\, T^m, \mathrm{tr}\, T^n\} = 0 \qquad T = \mathcal{P}\left[\exp\left(\int_{-s}^{s} A_x(t,y)\, \mathrm{d}y \right) \right] \tag{5.15}
$$

$(m, n) \in Z$. Since $\mathrm{tr}\, T^n$ are gauge invariant quantities, the previous relation holds also for the Abelian connections $A_{x,ab}^R$. It then follows that [8, 12]

$$
\{Q_m^R, Q_n^R\} = 0 \qquad \forall (n, m). \tag{5.16}
$$

Another set of conservation laws can be obtained in a completely analogous way by considering positive grade expansion in (5.4). The relevant equations are summarized in the appendix.

6. Conclusion

In this paper a bicomplex structure associated with the generalized Leznov–Saveliev equation is established. In this sense, the bicomplex structure is equivalent to the zero-curvature equation. Also, the linear problem associated with the zero-curvature condition is derived in terms of the bicomplex linear equation. The conservation laws for a non-Abelian Toda model were obtained, generalizing the standard procedure in the Abelian models.

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Appendix

Let

$$
A = \varepsilon^- + B^{-1}\partial B \qquad \bar{A} = -B^{-1}\varepsilon^+ B \qquad g_L = \left[\prod_{m=1}^{\infty} \exp(S_m) \right] \exp(\bar{\xi}\varepsilon^+)
$$

where S_m is a linear combination of generators in $\mathcal{F}^+$ and $\bar{\xi}(z, \bar{z})$ is a function. Under a gauge transformation

$$A^L = g_L A g_L^{-1} - \partial g_L g_L^{-1} = \sum_{m=-1}^{\infty} A^{L,m}$$
$$\bar{A}^L = g_L \bar{A} g_L^{-1} - \bar{\partial} g_L g_L^{-1} = \sum_{m=1}^{\infty} (\bar{A}_{\mathcal{A}}^{L,m} + \bar{A}_{\mathcal{F}}^{L,m})$$

where $A^{L,m} \in \hat{G}_m$, $\bar{A}_{\mathcal{A}}^{L,m}$ has grade m and is a linear combination of generators in $\mathcal{A}$. Similarly for $\bar{A}_{\mathcal{F}}^{L,m}$. Also,

$$\begin{aligned} B^{-1}\partial B = {} & \beta[\partial\varphi_1 - (3/2)\beta\tilde{\psi}\partial\tilde{\chi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}]H_{\lambda_1}^{(0)} + \beta[\partial\varphi_2 + (1/2)\beta\tilde{\psi}\partial\tilde{\chi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}]h_2^{(0)} \\ & + \beta[\partial\nu\hat{C} + \partial\eta\hat{D}] + \beta\partial\tilde{\chi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}E_{-\alpha_1}^{(0)} \\ & + \beta[\partial\tilde{\psi} + \beta\tilde{\psi}\partial(\varphi_1-\varphi_2) - \beta^2\tilde{\psi}^2\partial\tilde{\chi}\mathrm{e}^{\beta(\varphi_1-\varphi_2)}]E_{\alpha_1}^{(0)} \\ \equiv {} & J^{(1)}H_{\lambda_1}^{(0)} + J^{(2)}h_2^{(0)} + J^{\nu}\hat{C} + J^{\eta}\hat{D} + J^{-}E_{-\alpha_1}^{(0)} + J^{+}E_{\alpha_1}^{(0)}. \end{aligned}$$

Solving for $(S_m, \bar{\xi})$ such that

$$A^L = \varepsilon^- + J^\eta \hat{D}_{\mathcal{A}} + J^{(1)}H_{\lambda_1}^{(0)} + \sum_{m=1}^{\infty} a^{L,m}A_{(m)}$$

the zero-curvature equation leads to

$$\begin{aligned} & \bar{\partial}J^\eta = 0 \qquad \bar{\partial}J^{(1)} = 0 \qquad \bar{\partial}a^{L,1} = 0 \\ & \bar{A}_{\mathcal{F}}^{L,m} = 0 \qquad \forall m \\ & \partial\bar{a}_{\mathcal{A}}^{L,m} - \bar{\partial}a^{L,m} + \frac{m}{2}J^\eta\bar{a}_{\mathcal{A}}^{L,m} = 0 \qquad m \geqslant 2 \\ & \bar{a}_{\mathcal{A}}^{L,1} = 0 \end{aligned}$$

where $\bar{A}_{\mathcal{A}}^{L,m} = \bar{a}_{\mathcal{A}}^{L,m}A_{(m)}$.

Under another gauge transformation defined by

$$g_L^L = \exp\left[\varepsilon^- \int_L^z \mathrm{e}^{-\frac{1}{2}\int_L^w J^\eta(v)\,\mathrm{d}v}\,\mathrm{d}w\right]\exp\left[\int_L^z (J^{(1)}(w)H_{\lambda_1}^{(0)} + J^\eta(w)\hat{D}_{\mathcal{A}})\,\mathrm{d}w\right]$$

there follows

$$\bar{A}_{ab}^L \equiv \sum_{m=1}^{\infty}\bar{A}_{ab}^{L,m} = \sum_{m=1}^{\infty}\bar{a}_{\mathcal{A}}^{L,m}\mathrm{e}^{\frac{m}{2}\int_L^z J^\eta(v)\,\mathrm{d}v}A_{(m)} \qquad \partial\bar{A}_{ab}^L - \bar{\partial}A_{ab}^L = 0$$
$$\begin{aligned} A_{ab}^L \equiv \sum_{m=1}^{\infty}A_{ab}^{L,m} + b^{\hat{C}}\hat{C} = {} & \sum_{m=1}^{\infty}a^{L,m}\mathrm{e}^{\frac{m}{2}\int_L^z J^\eta(v)\,\mathrm{d}v}A_{(m)} - \mu_2 a^{L,1}\mathrm{e}^{\frac{1}{2}\int_L^z J^\eta(v)\,\mathrm{d}v} \\ & \times\int_L^z \mathrm{e}^{-\frac{1}{2}\int_L^w J^\eta(v)\,\mathrm{d}v}\mathrm{d}w\,\hat{C} \end{aligned}$$

where $L \in R$.

The conserved charges are obtained:

$$\begin{aligned} & \partial_t Q_m^L = 0 \qquad m \geqslant 1 \\ & Q_m^L = \int_{-s}^{s} A_{x,ab}^{L,m}(t,y)\,\mathrm{d}y \end{aligned}$$

where $s \in R$.

References

[1] Balog J, Feher L, Forgacs P, O Raifeartaigh L and Wipf A 1989 *Phys. Lett.* B **227** 214
Gomes J F, Sotkov G M and Zimerman A H 1999 *Ann. Phys., NY* **274** 289
(Gomes J F, Sotkov G M and Zimerman A H 1998 *Preprint* hep-th/9803234)

[2] Olive D I, Turok N and Underwood J W R 1993 *Nucl. Phys.* B **401** 663
Lund F and Regge T 1976 *Phys. Rev.* D **14** 1524
Castro Alvaredo O A and Miramontes J L 2000 *Nucl. Phys.* B **581** 643
(Castro Alvaredo O A and Miramontes J L 2000 *Preprint* hep-th/0002219)

[3] Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2001 *Nucl. Phys.* B **598** 615
(Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2000 *Preprint* hep-th/0011187)
see also
Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2000 *Nucl. Phys.* B at press
(Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2000 *Preprint* hep-th/0007169)

[4] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1991 *Phys. Lett.* B **254** 372
Babelon O and Bonora L 1990 *Phys. Lett.* B **244** 220

[5] Leznov A N and Saveliev M V 1983 *Commun. Math. Phys.* **89** 59
Leznov A N and Saveliev M V 1983 *Theor. Math. Phys.* **54** 209

[6] Faddeev L D 1984 *Les Houches Lectures, Session XXXIX* ed J B Zuber and R Stora (Amsterdam: Elsevier)
Faddeev L D and Takhtajan L A 1987 *Hamiltonian Methods in the Theory of Solitons* (Berlin: Springer)

[7] Dimakis A and Muller-Hoissen F 2000 *Int. J. Mod. Phys.* B **14** 2455
(Dimakis A and Muller-Hoissen F 2000 *Preprint* hep-th/0006005)
Dimakis A and Muller-Hoissen F 2000 *J. Phys. A: Math. Gen.* **33** 6579
(Dimakis A and Muller-Hoissen F 2000 *Preprint* hep-th/0006029)
Dimakis A and Muller-Hoissen F 2000 *Preprint* hep-th/0007015
Dimakis A and Muller-Hoissen F 2000 *Phys. Lett.* A **278** 139
(Dimakis A and Muller-Hoissen F 2000 *Preprint* hep-th/0007074)
Dimakis A and Muller-Hoissen F 2000 *J. Phys. A: Math. Gen.* **34** 2571
(Dimakis A and Muller-Hoissen F 2000 *Preprint* nlin.SI/0008016)

[8] Olive D I and Turok N 1985 *Nucl. Phys.* B **257** [FS14] 277

[9] Goddard P and Olive D I 1986 *Int. J. Mod. Phys.* A **1** 303
Kac V G 1990 *Infinite-Dimensional Lie Algebras* (Cambridge: Cambridge University Press)

[10] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1998 *Supersymmetry and Integrable Models (Lecture Notes in Physics vol 502)* ed H Aratyn *et al* (Berlin: Springer) p 197

[11] Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2000 *Proc. 23rd Int. Coll. on Group Theoretical Methods in Physics (Dubna, 2000)* ed G Pogosyan *et al* at press
(Gomes J F, Gueuvoghlanian E P, Sotkov G M and Zimerman A H 2000 *Preprint* hep-th/0010257)

[12] Aratyn H, Ferreira L A, Gomes J F and Zimerman A H 1994 *Mod. Phys. Lett.* A **9** 2783